SOCP Relaxation of Sensor Network Localization

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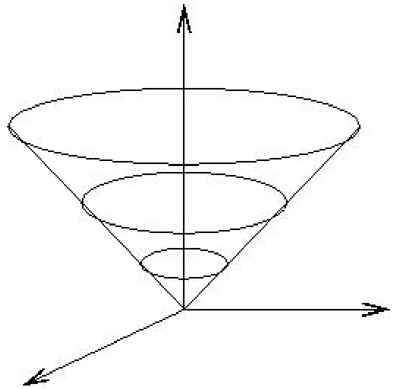
Talk Outline

- Problem description
- SDP and SOCP relaxations
- Properties of SDP and SOCP relaxations
- Performance of SOCP relaxation and efficient solution methods
- Conclusions & Future Directions



Sensor Network Localization

Basic Problem:

- n pts in $\Re^d$ ($d = 1, 2, 3$).
- Know last $n - m$ pts ('anchors') $x_{m+1}, ..., x_n$ and Eucl. dist. estimate for pairs of 'neighboring' pts

$$d_{ij} \geq 0 \quad \forall (i, j) \in \mathcal{A}$$

with $\mathcal{A} \subseteq \{(i, j) : 1 \leq i < j \leq n\}$.

- Estimate first m pts ('sensors').

History? Graph realization, position estimation in wireless sensor network, determining protein structures, ...

Optimization Problem Formulation

$$v_{\text{opt}} := \min_{x_1,\ldots,x_m} \sum_{(i,j)\in\mathcal{A}} \left| \|x_i - x_j\|^2 - d_{ij}^2 \right|$$

- Objective function is nonconvex.
- Problem is NP-hard (reduction from PARTITION).
- Use a convex (SDP, SOCP) relaxation.

SDP and SOCP

$$\begin{array}{ll} \min & \langle c, z\rangle \\ \text{s.t.} & \langle a^k, z\rangle = b^k, \quad k = 1,\ldots,p \\ & z \succeq 0 \end{array}$$

LP: $z = [\, z_i \,]_{i=1}^q \in \Re^q$ $\quad \langle c, z\rangle = c^T z$ $\quad z \succeq 0 \;\Leftrightarrow\; z_i \geq 0 \;\forall i$

SDP: $z = [\, z_{ij} \,]_{i,j=1}^q \in \Re^{q\times q}$ sym. $\quad \langle c, z\rangle = \mathrm{tr}(c^T z)$ $\quad z \succeq 0 \;\Leftrightarrow\; \lambda_i(z) \geq 0 \;\forall i$

SOCP: $z \in \Re^q$ $\quad \langle c, z\rangle = c^T z$ $\quad z \succeq 0 \;\Leftrightarrow\; z \in K^{q_1} \times \cdots \times K^{q_N}$

$$K^q = \left\{ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \Re \times \Re^{q-1} : \|z_2\| \leq z_1 \right\}$$

LP solvable by simplex or interior-pt methods. SDP, SOCP solvable by interior-pt methods.

SDP Relaxation

Let $X := [x_1 \ \cdots \ x_m], \quad A := [x_{m+1} \ \cdots \ x_n]$.

Then (Biswas,Ye '03)

$$\|x_i - x_j\|^2 \quad = \quad \operatorname{tr}\left(b_{ij} b_{ij}^T \begin{bmatrix} X^T X & X^T \\ X & I_d \end{bmatrix} \right)$$

with $b_{ij} := \begin{bmatrix} I_m & 0 \\ 0 & A \end{bmatrix} (e_i - e_j)$.

Fact: $\begin{bmatrix} Y & X^T \\ X & I_d \end{bmatrix} \succeq 0$ has rank $d \quad \Leftrightarrow \quad Y = X^T X$

Thus

$$\begin{aligned} \upsilon_{\text{opt}} = \quad & \min_{X,Y} \quad \sum_{(i,j)\in\mathcal{A}} \left|\text{tr}\left(b_{ij}b_{ij}^T Z\right) - d_{ij}^2\right| \\ & \text{s.t.} \quad Z = \begin{bmatrix} Y & X^T \\ X & I_d \end{bmatrix} \succeq 0, \quad \text{rank} Z = d \end{aligned}$$

Drop low-rank constraint:

$$\begin{aligned} \upsilon_{\text{sdp}} := \quad & \min_{X,Y} \quad \sum_{(i,j)\in\mathcal{A}} \left|\text{tr}\left(b_{ij}b_{ij}^T Z\right) - d_{ij}^2\right| \\ & \text{s.t.} \quad Z = \begin{bmatrix} Y & X^T \\ X & I_d \end{bmatrix} \succeq 0 \end{aligned}$$

- Biswas and Ye gave probabilistic interpretation of SDP soln, and proposed a distributed (domain partitioning) method for solving SDP when $n > 100$. Refine soln by gradient search (Liang, Wang, Ye '04)..

SOCP Relaxation

Second-order cone program (SOCP) is easier to solve than SDP.

- Q: Is SOCP relaxation a good approximation? Or a mixed SDP-SOCP relaxation?
- Q: How to efficiently solve SOCP relaxation?

SOCP Relaxation

$$\begin{aligned} v_{\text{opt}} = \quad & \min_{x_1,\ldots,x_m,y_{ij}} \quad \sum_{(i,j)\in\mathcal{A}} \left|y_{ij} - d_{ij}^2\right| \\ & \text{s.t.} \quad y_{ij} = \|x_i - x_j\|^2 \quad \forall (i,j) \in \mathcal{A} \end{aligned}$$

Relax "$=$" to "$\geq$" constraint:

$$\begin{aligned} v_{\text{socp}} = \quad & \min_{x_1,\ldots,x_m,y_{ij}} \quad \sum_{(i,j)\in\mathcal{A}} \left|y_{ij} - d_{ij}^2\right| \\ & \text{s.t.} \qquad y_{ij} \geq \|x_i - x_j\|^2 \quad \forall (i,j) \in \mathcal{A} \end{aligned}$$

$$y \geq \|x\|^2 \quad \Leftrightarrow \quad y + 1 \geq \|(y-1, 2x)\| \quad \Leftrightarrow \quad (y+1, y-1, 2x) \in K^{d+2}$$

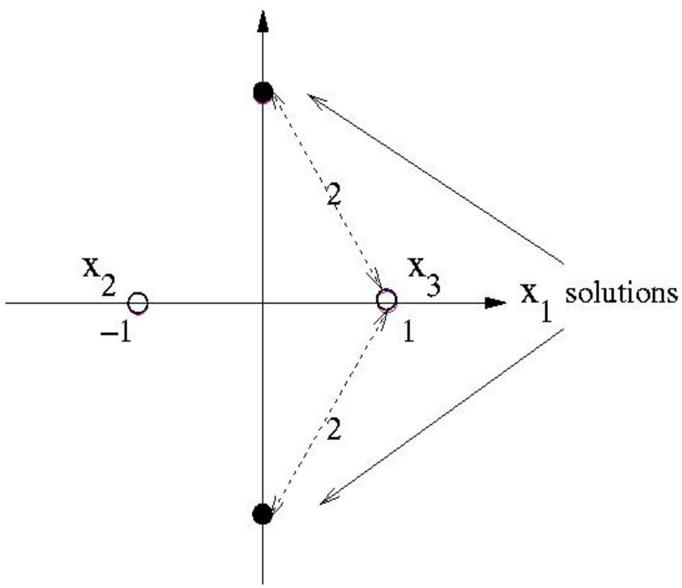
(also Doherty,Pister,El Ghaoui '03)

Properties of SDP, SOCP Relaxations

$d = 2$, $n = 3$, $m = 1$, $d_{12} = d_{13} = 2$

Problem:

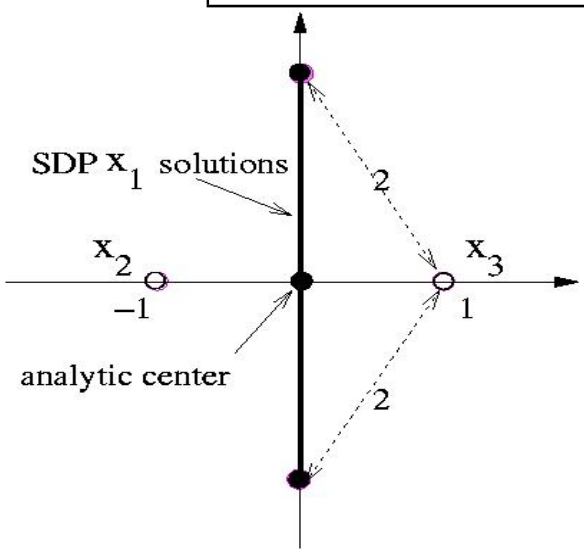
$$0 = \min_{x_1=(\alpha,\beta)\in\Re^2} \quad |(1-\alpha)^2 + \beta^2 - 4| + |(-1-\alpha)^2 + \beta^2 - 4|$$



SDP Relaxation:

$$0 = \min_{\substack{x_1=(\alpha,\beta)\in\Re^2\\ y\in\Re}} |y-2\alpha-3| + |y+2\alpha-3|$$

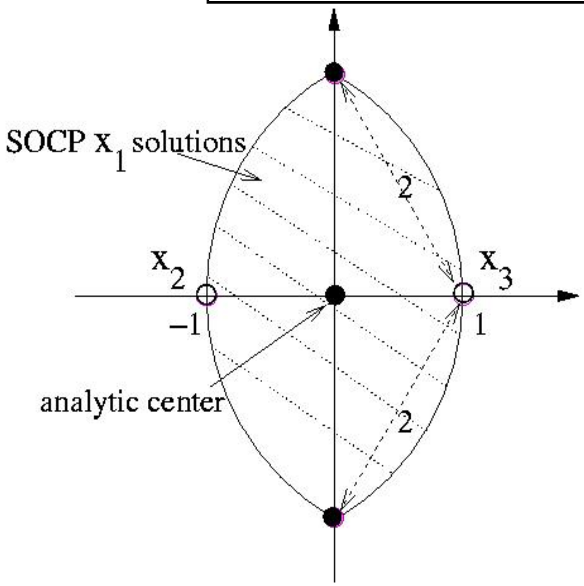
$$\text{s.t.} \quad \begin{bmatrix} y & \alpha & \beta \\ \alpha & 1 & 0 \\ \beta & 0 & 1 \end{bmatrix} \succeq 0$$



If solve SDP by IP method, then likely get analy. center of soln set.

SOCP Relaxation:

$$
\begin{aligned}
0 = \quad & \min_{\substack{x_1=(\alpha,\beta)\in\Re^2 \\ y,z\in\Re}} |y-4| + |z-4| \\
\text{s.t.} \quad & y \geq (1-\alpha)^2 + \beta^2 \\
& z \geq (-1-\alpha)^2 + \beta^2
\end{aligned}
$$



If solve SOCP by IP method, then likely get analy. center of soln set.

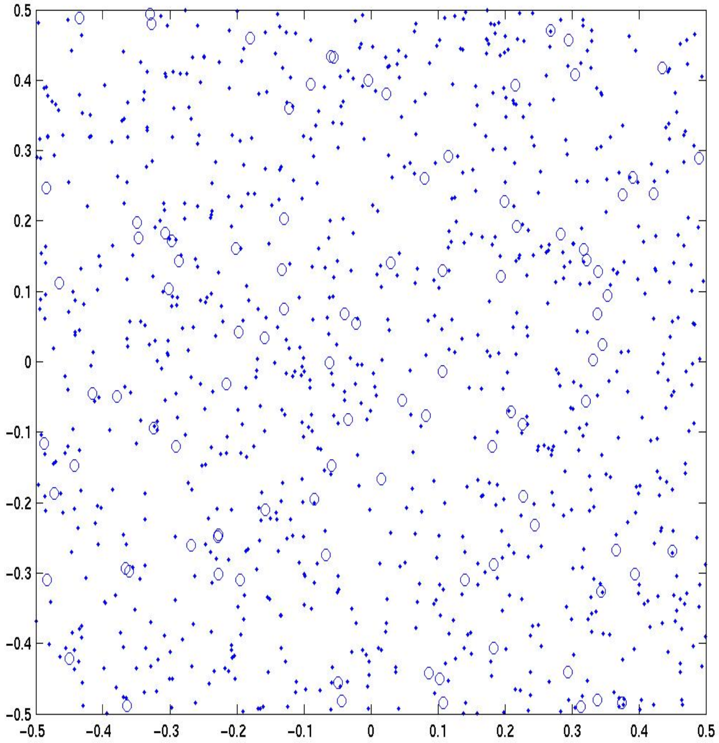
Properties of SDP & SOCP Relaxations

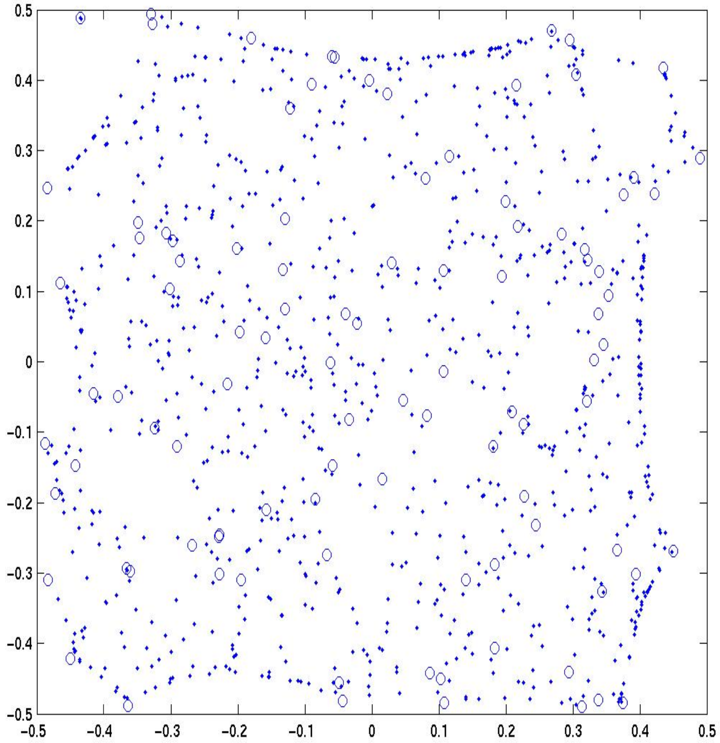
Fact 1: $v_{\mathrm{socp}} \le v_{\mathrm{sdp}}$. If $v_{\mathrm{socp}} = v_{\mathrm{sdp}}$, then
$\{$SOCP $(x_1, ..., x_m)$ solns$\} \supseteq \{$SDP $(x_1, ..., x_m)$ solns$\}$.

Fact 2: If $(x_1, ..., x_m, y_{ij})_{(i,j)\in\mathcal{A}}$ is the analytic center soln of SOCP, then

$$x_i \in \mathrm{conv}\,\{x_j\}_{j\in\mathcal{N}(i)} \quad \forall i \le m$$

with $\mathcal{N}(i) := \{j \ : \ (i,j) \in \mathcal{A}\}$.





True soln ($m = 900$, $n = 1000$, nhbrs if dist$< .06$)

SOCP soln found by IP method (SeDuMi)

Fact 3: If $X = [x_1 \ \cdots \ x_m], Y$ is a relative-interior SDP soln (e.g., analytic center), then for each i,

$$\|x_i\|^2 = Y_{ii} \quad \Longrightarrow \quad x_i \text{ appears in every SDP soln.}$$

Fact 4: If $(x_1, ..., x_m, y_{ij})_{(i,j)\in\mathcal{A}}$ is a relative-interior SOCP soln (e.g., analytic center), then for each i,

$$\|x_i - x_j\|^2 = y_{ij} \quad \text{for some } j \in \mathcal{N}(i) \quad \Leftrightarrow \quad x_i \text{ appears in every SOCP soln.}$$

Error Bounds

What if distances have errors?

$$d_{ij}^2 = \bar{d}_{ij}^2 + \delta_{ij},$$

where $\delta_{ij} \in \Re$ and $\bar{d}_{ij} := \|x_i^{\text{true}} - x_j^{\text{true}}\|$ ($x_i^{\text{true}} = x_i \ \forall\, i > m$).

Fact 5: If $(x_1, ..., x_m, y_{ij})_{(i,j)\in\mathcal{A}}$ is a relative-interior SOCP soln corresp. $(d_{ij})_{(i,j)\in\mathcal{A}}$ and $\sum_{(i,j)\in\mathcal{A}} |\delta_{ij}| \leq \delta$, then for each i,

$$\|x_i - x_j\|^2 = y_{ij} \quad \text{for some } j \in \mathcal{N}(i) \quad \Longrightarrow \quad \|x_i - x_i^{\text{true}}\| = O\Big(\sqrt{\sum_{(i,j)\in\mathcal{A}} |\delta_{ij}|}\Big).$$

Fact 6: As $\sum_{(i,j)\in\mathcal{A}} |\delta_{ij}| \to 0$, (analytic center SOCP soln corresp. $(d_{ij})_{(i,j)\in\mathcal{A}}$) $\to$ (analytic center SOCP soln corresp. $(\bar{d}_{ij})_{(i,j)\in\mathcal{A}}$).

Solving SOCP Relaxation I: IP Method

$$\begin{array}{ll} \min_{x_1,\ldots,x_m,y_{ij}} & \sum_{(i,j)\in\mathcal{A}} \left| y_{ij} - d_{ij}^2 \right| \\ \text{s.t.} & y_{ij} \geq \|x_i - x_j\|^2 \quad \forall (i,j) \in \mathcal{A} \end{array}$$

Put into standard SOCP form:

$$\begin{array}{llll} \min & \sum_{(i,j)\in\mathcal{A}} u_{ij} & & \\ \text{s.t.} & x_i - x_j - w_{ij} & = 0 & \forall (i,j) \in \mathcal{A} \\ & y_{ij} - u_{ij} & = d_{ij}^2 & \forall (i,j) \in \mathcal{A} \\ & \alpha_{ij} & = \frac{1}{2} & \forall (i,j) \in \mathcal{A} \\ & u_{ij} \geq 0, & (\alpha_{ij}, y_{ij}, w_{ij}) \in \mathrm{Rcone}^{d+2} & \forall (i,j) \in \mathcal{A} \end{array}$$

with $\mathrm{Rcone}^{d+2} := \{(\alpha, y, w) \in \Re \times \Re \times \Re^d : \|w\|^2/2 \leq \alpha y\}$.

Solve by an IP method, e.g., SeDuMi 1.05 (Sturm '01).

Solving SOCP Relaxation II: Smoothing + Coordinate Gradient Descent

$$\min_{y \geq z} |y - d^2| = \max\{0, z - d^2\}$$

So SOCP relaxation:

$$\min_{x_1,\ldots,x_m} \quad \sum_{(i,j)\in\mathcal{A}} \max\{0, \|x_i - x_j\|^2 - d_{ij}^2\}$$

This is an unconstrained nonsmooth convex program.

- Smooth approximation:

$$\max\{0, t\} \;\approx\; \mu h(t/\mu) \quad (\mu > 0)$$

h smooth convex, $\lim_{t\to-\infty} h(t) = \lim_{t\to\infty} h(t) - t = 0$. We use $h(t) = ((t^2 + 4)^{1/2} + t)/2$ (CHKS).

SOCP approximation:

$$\min \quad f_\mu(x_1,..,x_m) := \sum_{(i,j)\in\mathcal{A}} \mu h\left(\frac{\|x_i - x_j\|^2 - d_{ij}^2}{\mu}\right)$$

Add a smoothed log-barrier term $-\mu \sum_{(i,j)\in\mathcal{A}} \log\left(\mu h\left(\frac{d_{ij}^2 - \|x_i - x_j\|^2}{\mu}\right)\right)$

Solve the smooth approximation using coordinate gradient descent (SCGD):

- If $\|\nabla_{x_i} f_\mu\| = \Omega(\mu)$, then update x_i by moving it along the Newton direction $-[\nabla^2_{x_i x_i} f_\mu]^{-1} \nabla_{x_i} f_\mu$, with Armijo stepsize rule, and re-iterate.
- Decrease μ when $\|\nabla_{x_i} f_\mu\| = O(\mu)\ \forall i$.

$\mu^{\text{init}} = 1e-5$. $\mu^{\text{end}} = 1e-9$. Decrease μ by a factor of 10.

Code in Fortran. Computation easily distributes.

Simulation Results

- Uniformly generate $x_1^{\text{true}}, ..., x_n^{\text{true}}$ in $[0,1]^2$, $m = .9n$,
  two pts are nhbrs if dist$< \ radioR$.
  Set
  $$d_{ij} = \|x_i^{\text{true}} - x_j^{\text{true}}\| \cdot |1 + \epsilon_{ij} \cdot nf|,$$
  $\epsilon_{ij} \sim N(0,1)$ (Biswas, Ye '03)

- Solve SOCP using SeDuMi 1.05 (1.1?) or SCGD.

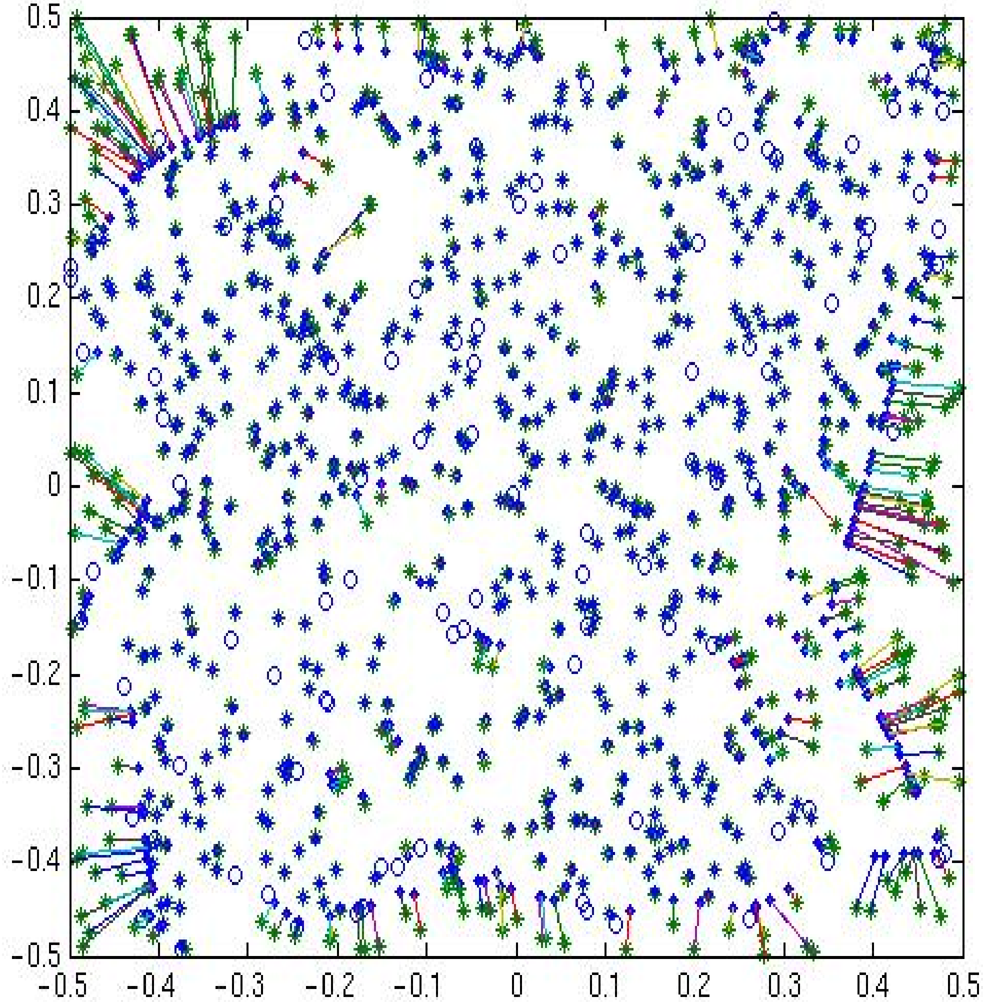
- Sensor i is uniquely positioned if

$$\left| \|x_i - x_j\|^2 - y_{ij} \right| \leq 10^{-7} d_{ij} \quad \text{for some } j \in \mathcal{N}(i).$$

n	m	nf	SeDuMi cpu/$m_{\rm up}$/Err$_{\rm up}$	SCGD cpu/$m_{\rm up}$/Err$_{\rm up}$
1000	900	0	3.6/402/7.2e-4	.2/357/3.8e-5
1000	900	.001	3.2/473/1.8e-3	.4/442/1.5e-3
1000	900	.01	3.9/554/1.5e-2	1.6/518/1.1e-2
2000	1800	0	176.6/1534/4.3e-4	0.8/1541/3.3e-4
2000	1800	.001	208.6/1464/3.6e-3	1.8/1466/3.6e-3
2000	1800	.01	161.8/1710/5.1e-2	2.9/1707/5.1e-2
4000	3600	0	202.5/2851/4.0e-4	1.6/2844/3.2e-4
4000	3600	.001	193.8/2938/3.2e-3	5.1/2894/3.0e-3
4000	3600	.01	196.3/3073/1.0e-2	6.1/3020/9.1e-3

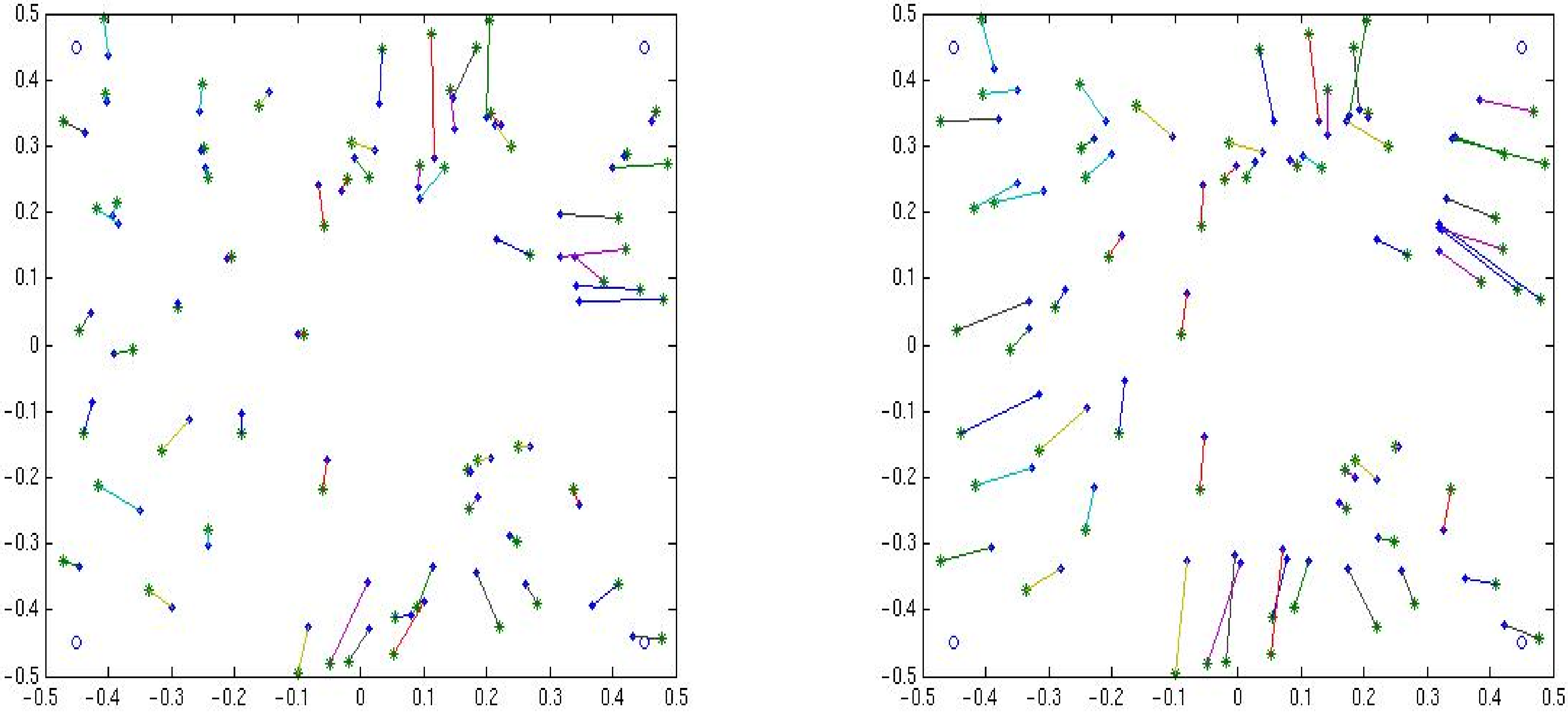
Table 1: $radioR = .06(.035)$ for $n = 1000, 2000(4000)$

- cpu (minutes) times are on a HP DL360 workstation, running Linux 3.5.
- $m_{\rm up} :=$ number of uniquely positioned sensors.
- Err$_{\rm up} := \max_{i \text{ uniq. pos.}} \|x_i - x_i^{\rm true}\|$.



○ = anchors ∗ = true positions of sensors
• = SOCP soln found by SCGD ($m = 900$, $n = 1000$, $radioR = .06$, $nf = .01$)

Comparing SDP and SOCP analytic center solns on small noisy problem.



$\circ$ = anchors $\quad\quad$ $*$ = true positions of sensors

$\bullet$ = SDP soln found by SeDuMi (left), SOCP soln found by SCGD (right).

($m = 60$, $n = 64$, $radioR = .3$, $nf = .1$)

Mixed SDP-SOCP Relaxation

Choose $0 \leq \ell \leq m$. Let $\mathcal{B} := \{(i,j) \in \mathcal{A} \ : \ i \leq \ell, j \leq \ell\}$.

$$\begin{array}{lll} \min\limits_{x_1,\ldots,x_m,y_{ij},Y} & \sum\limits_{(i,j)\in\mathcal{B}} \left|\mathrm{tr}\left(b_{ij}b_{ij}^T Z\right) - d_{ij}^2\right| + \sum\limits_{(i,j)\in\mathcal{A}\setminus\mathcal{B}} \left|y_{ij} - d_{ij}^2\right| \\ \text{s.t.} & Z = \begin{bmatrix} Y & X^T \\ X & I_d \end{bmatrix} \succeq 0 \\ & X = [x_1 \ \cdots \ x_\ell] \\ & y_{ij} \geq \|x_i - x_j\|^2 \quad \forall (i,j) \in \mathcal{A} \setminus \mathcal{B} \end{array}$$

with $b_{ij} := \begin{bmatrix} I_\ell & 0 & 0 \\ 0 & 0 & A \end{bmatrix} (e_i - e_j)$.

Easier to solve than SDP? As good a relaxation? Properties?

Conclusions & Future Directions

- SOCP relaxation may be a good pre-processor. Relative interior soln is key.
- Faster methods for solving SOCP? Exploit network structures of SOCP? (For $d = 1$, solvable by ϵ-relaxation method (Bertsekas,Polymenakos,T '97))
- Error bound for SDP relaxation?
- Additional (convex) constraints? Other objective functions, e.g.,
  $\sum_{(i,j)\in\mathcal{A}} \left| \|x_i - x_j\| - d_{ij} \right|^2$?
- Replace 2-norm by a p-norm ($1 \le p \le \infty$) $\Longrightarrow$ p-order cone relaxation.

  Q: For $x_1, ..., x_n \in \Re^d$, is $\arg\min_x \sum_{i=1}^n \|x - x_i\|_p^p \in \mathrm{conv}\,\{x_1, ..., x_n\}$?
  ($1 < p < \infty$)
  A: Yes for $d \le 2$. No for $d \ge 3$.