Pinning Control of Complex Networks

GUANRONG CHEN

Centre for Chaos and Complex Networks
City University of Hong Kong, P. R. China

Acknowledgements

Zhengping Fan
Xiaofan Wang
Xiang Li

OUTLINE

- **Introduction**
- **Control of Scale-Free Networks**
  - **Pinning:** **Randomly Selective Control**
  - **Pinning:** **Specially Selective Control**
  - **Comparisons**
- **Conclusions**

Introduction

Control (electronic or mechanical devices, stimuli, policy or commands, …)

For **a single node** (a single system): What kind of controller to use? How to design it?

For **a network of nodes**: (In addition), How many nodes to control? Which nodes to control? – Network topology matters

Introduction

Control of Networks

- → Network Synchronization
- → Network Stabilization
- → Network Utilization
- → Networked Sensing
- → Networked Controlling
- → ……

Introduction

Pinning Control of Networks

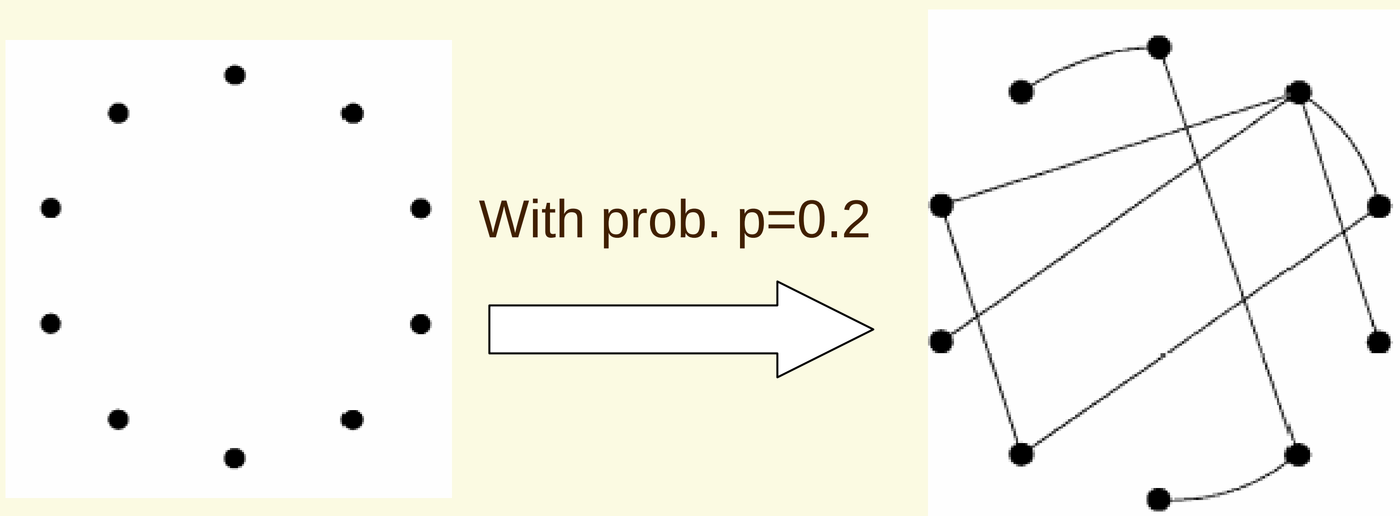
For **a network of nodes**: How many nodes to control? Which nodes to control? – to pin (a controller will not be removed after being placed in)

- Random-Graph Networks
- Scale-Free Networks

Random Graph Model

(Erdös-Rényi: 1960)

- ✓ Start with *N* **nodes** and no **links**
- ✓ With probability *p*, connect two randomly selected **nodes** with a **link**



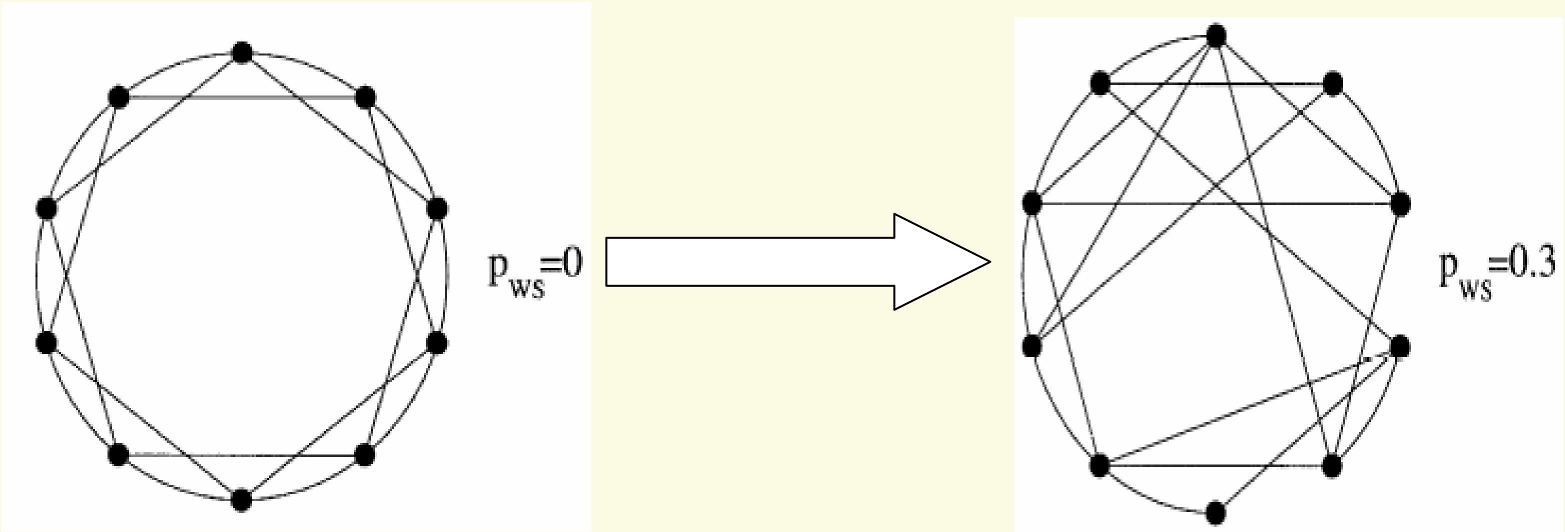
Small-World Network Model

(Watts-Strogatz:1998)

- **Start with a lattice of *N* nodes with links between the nearest and next-nearest neighbors**
- **Each link is rewired with probability *p***

Here, rewiring means shifting one end of a link to a randomly selected node

Small-World Network Model



Random Graph Model and Small-World Network Model

Some Common Features:

- **Connectivity Distribution:**
  Poisson/binomial or near uniform distribution
- **Homogeneous Nature:**
  Each node has roughly the same number of links
- **Network Size:**
  Network does not grow

Scale-free Network Model

(Barabasi-Albert:1999)

Features:

- **Connectivity Distribution: power-law distribution**
  $\sim k^{-r}$ with $r = 3$

- **Non-homogeneous Nature:**
  A few nodes have many links but most other nodes only have a few links

- **Network Size:**
  Network continuously grows

Extended BA (EBA) Model (allows $r < 3$)

(Albert and Barabasi: 2000)

Extended BA (EBA) Model

The EBA model (Albert and Barabasi: 2000) --

(i) Add new links between existing nodes:

With probability p, m $(m \le m_0)$ new links are added into the network: one end of each link is chosen at random, and the other end is selected with probability

$$\Pi(k_i) = \frac{k_i + 1}{\sum_l (k_l + 1)}$$

EBA Model

(ii) Re-wiring: With probability q, m links are rewired: First, a node i with a link l_{ij} is selected at random. Then, this link is replaced with a new link $l_{ij'}$ that connects node i to node j' which is chosen with probability $\Pi(k_{j'})$

(iii) Incremental growth: With probability $1-p-q$, a new node is added into the network: The new node has m new links to the already existing nodes in the network with probability $\Pi(k_i)$.

EBA Model

In this model, $0 \le p < 1$ and $0 \le q < 1 - p$.

If $q < \min(1 - p, (1 - p + m)/(1 + 2m))$, then the connectivity distribution of nodes will be in a **power-law** form:

$$P(k) \propto (k + A(p,q,m) + 1)^{-\gamma}$$

where $\gamma = 1 + B$.

$$A(p,q,m) = (p - q)\left(\frac{2m(1-q)}{1-p-q} + 1\right)$$

$$B(p,q,m) = \frac{2m(1-q) + 1 - p - q}{m}$$

A Typical Model of Scale-Free Networks

A network with N linearly coupled nodes:

$$\dot{x}_i = f(x_i) + c\sum_{\substack{j=1 \\ j\neq i}}^{N} a_{ij}\Gamma(x_j - x_i), i = 1,2,\cdots,N \qquad (1)$$

Here:

$x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in R^n$ - state vectors

$f(\cdot)$ - nonlinear function

$\Gamma \in R^{n\times n}$ - constant 0-1 coupling matrix

Assume: $\Gamma = diag(r_1, \cdots, r_n)$ is diagonal with $r_i = 1$ for a particular i, and $r_j = 0$ for $j \neq i$

A Typical Model of Scale-Free Networks

Let the constant coupling strength be $c > 0$.

If there is a link between node i and node j ($j \neq i$), then let $a_{ij} = a_{ji} = 1$; otherwise, let

$$a_{ij} = a_{ji} = 0 \quad (i \neq j)$$

Define

$$\sum_{\substack{j=1 \\ j \neq i}}^{N} a_{ij} = \sum_{\substack{j=1 \\ j \neq i}}^{N} a_{ji} = k_i, \qquad i = 1, 2, \cdots, N$$

and let

$$a_{ii} = -k_i \qquad (i = 1, 2, \cdots, N)$$

A Typical Model of Scale-Free Networks

Model (1) can be rewritten as

$$\dot{x}_i = f(x_i) + c\sum_{j=1}^{N} a_{ij}\Gamma x_j \qquad i = 1, 2, \cdots, N \qquad (2)$$

Here, the coupling matrix $A = (a_{ij}) \in R^{N \times N}$ represents the coupling configuration of the entire network.

Assume: $A = (a_{ij})_{N \times N}$ is a symmetric and irreducible matrix.

Then, λ_1 , the largest eigenvalue of the matrix A, is zero, with multiplicity 1, and all the other eigenvalues are strictly negative: $\lambda_N \le \cdots \le \lambda_2 < 0$

[C. W. Wu: *Synchronization in Coupled Chaotic Circuits and Systems*, World Scientific, 2002]

Control of Scale-Free Networks

Here, the control objective is:

To stabilize network (2) onto a particular solution of the network:

$$x_1(t) = x_2(t) = \cdots = x_N(t) \rightarrow \bar{x}, \quad as \quad t \rightarrow \infty$$

Here, $\bar{x} \in R^n$ is an equilibrium point of an isolated node.

(For example, if the network is not synchronizable, then control is needed.)

Control of Scale-Free Networks

- It is very difficult, if not impossible, to control every node in a very large-scale complex dynamical network

- Even if it is possible, the cost would be very high

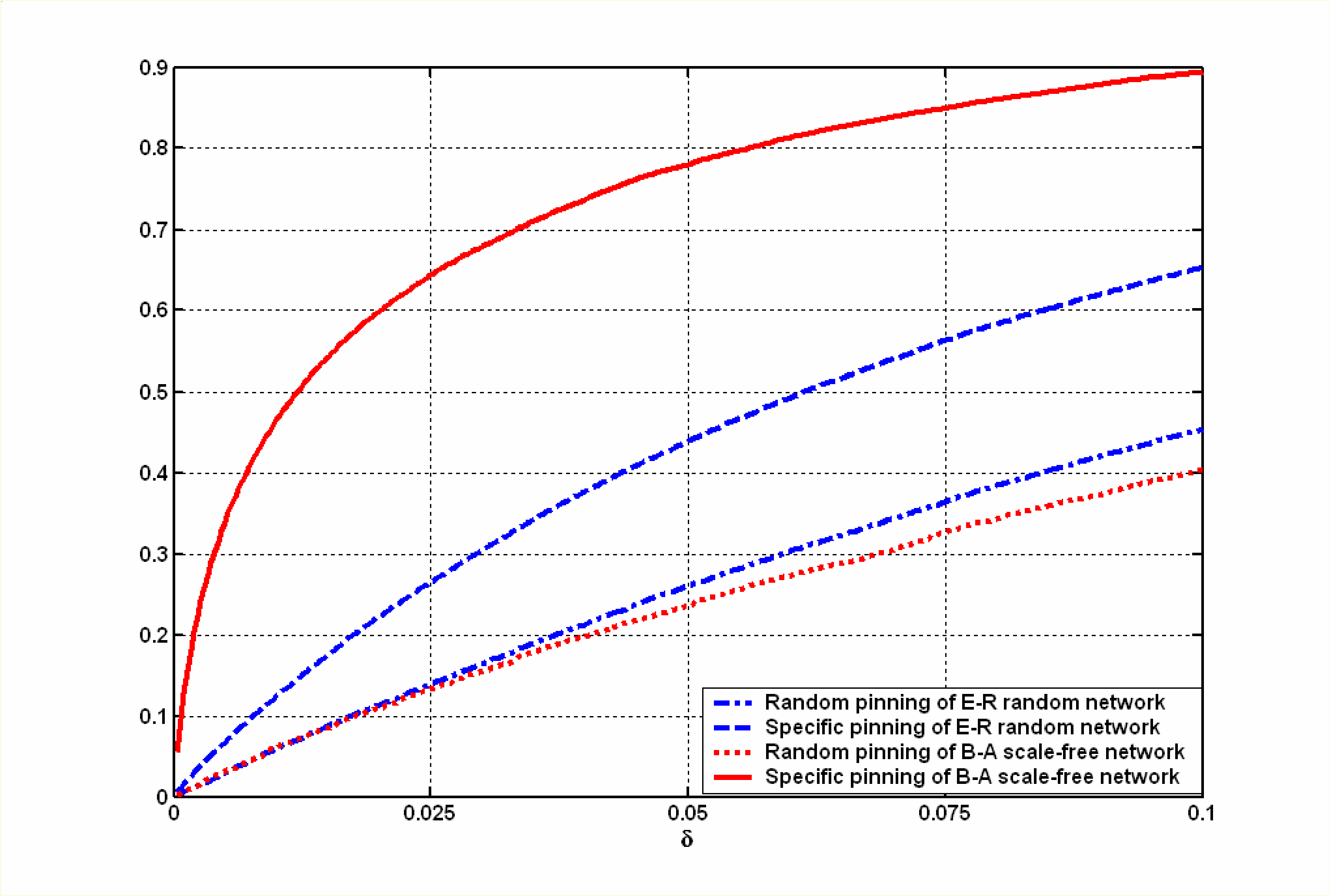
Pinning Control:

Only a small portion of nodes are selected to apply control

1. Decentralized pinning control
2. Selective pinning control

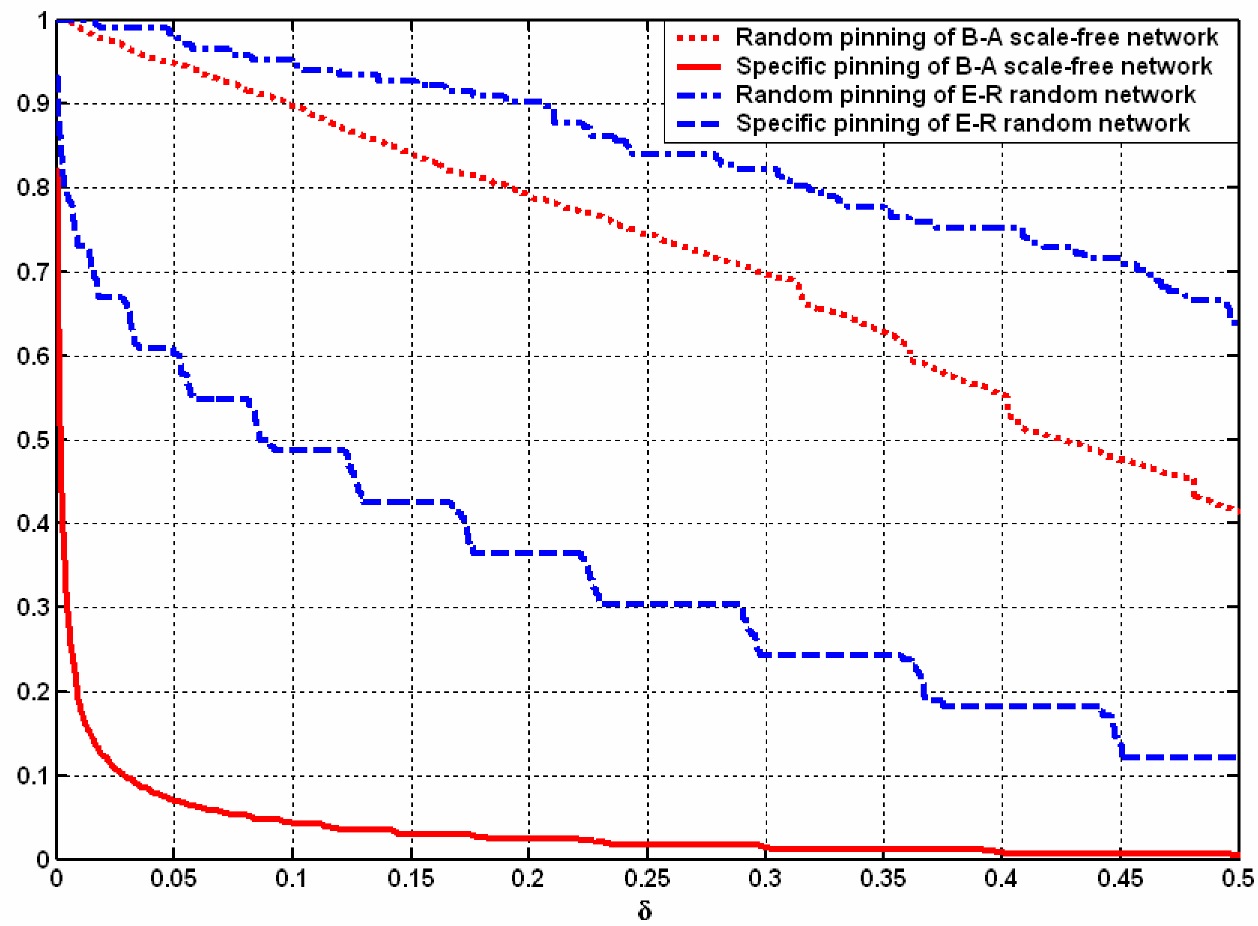
[X. F. Wang and G. Chen, Physica A, 2002, 310: 521-531]
[X. Li, X. F. Wang and G. Chen, IEEE Trans. CAS-I: 2004, 51(10): 2074-2087]

Pinning Control: A Comparison (stabilization)



Percentage of nodes affected by pinning control in the network of 3000 nodes

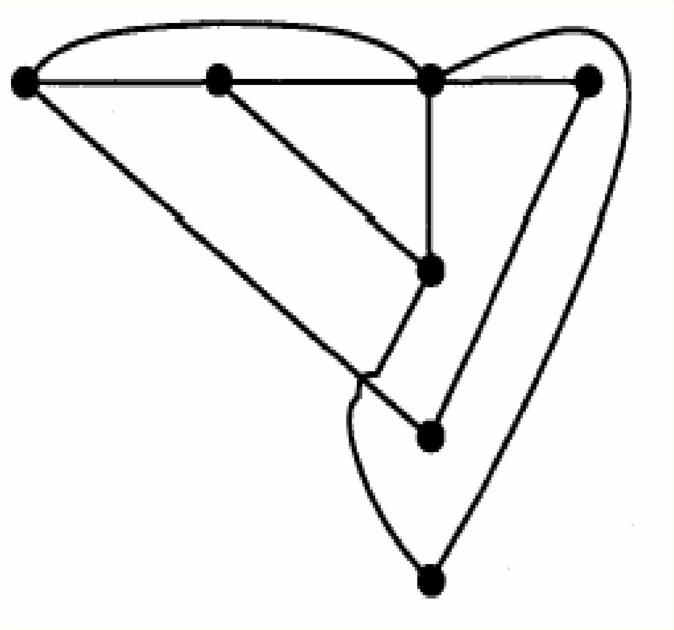
Pinning Control: A Comparison (attack)

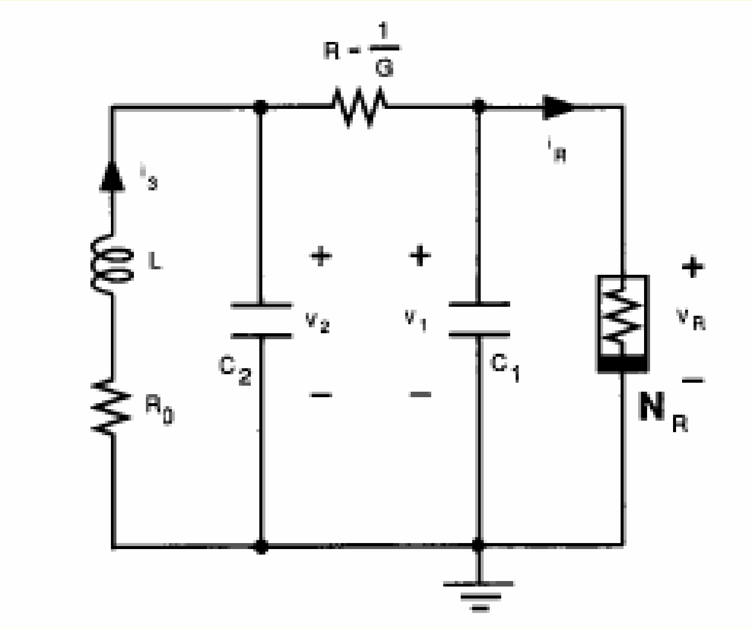


Percentage of remaining connectivity in the network of 3000 nodes

Pinning Control of Scale-Free Networks:

Example: Networked Chua's circuits





Chua Circuit

[C. W. Wu and L. O. Chua: IEEE Trans. CAS-I, 1995, 494-497]

[C. W. Wu: *Synchronization in Coupled Chaotic Circuits and Systems,* 2002, World Scientific]

Pinning Control of Scale-Free Networks

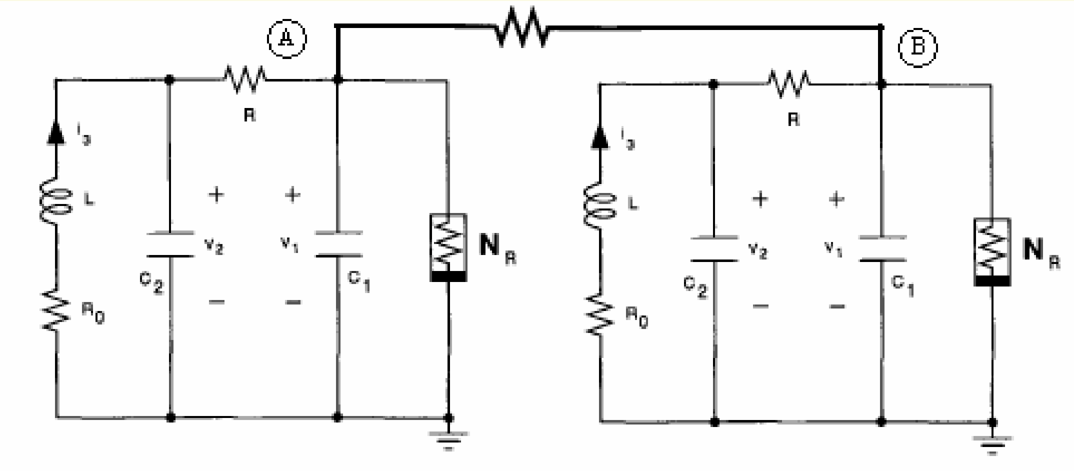
Chua's Circuit:

$$\begin{cases} \dfrac{dv_1}{dt} = \dfrac{1}{C_1}\left[G(v_2 - v_1) - f(v_1)\right] \\ \dfrac{dv_2}{dt} = \dfrac{1}{C_2}\left[G(v_1 - v_2) + i_3\right] \\ \dfrac{di_3}{dt} = -\dfrac{1}{L}\left[v_2 + R_0 i_3\right] \end{cases}$$

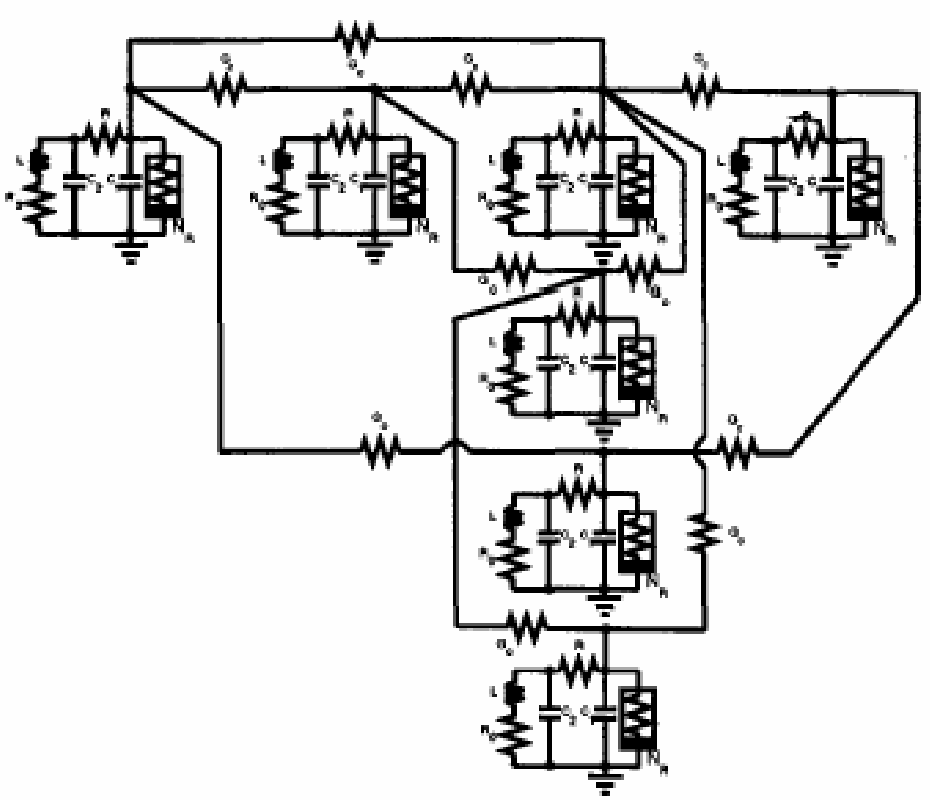
$$f(v_1) = G_b v_1 + \frac{1}{2}(G_a - G_b)\left\{|v_1 + E| - |v_1 - E|\right\}$$

Pinning Control of Scale-Free Networks

If there exists a link between node *A* and *B*,
then they will be coupled by a linear resistor:



Pinning Control of Scale-Free Networks



The coupled network of Chua's circuits

[C. W. Wu and L. O. Chua: IEEE Trans. CAS-I, 1995, 494-497]

Pinning Control of Chua-Circuit Network

BA scale-free network of Chua's circuits:

$$\begin{cases} dx_i / dt = \alpha(y_i - x_i - f(x_i)) + c\sum_{j=1}^{N} a_{ij} x_j + u_i \\ dy_i / dt = x_i - y_i + z_i \\ dz_i / dt = -\beta y_i \end{cases} \qquad (i = 1, \ldots, N)$$

where

$$f(x_i) = bx_i + \frac{1}{2}(d - b)\left(|x_i + 1| - |x_i - 1|\right)$$

$$u_i = -kx_i \ (i = 1, \ldots, N)$$

(state feedback controller)

Pinning Control of Chua-Circuit Network

Circuit parameters:

$$\alpha = 9.78, \beta = 14.97, b = -0.75, d = -1.3$$

Network parameters:

Network size: $N = 200$

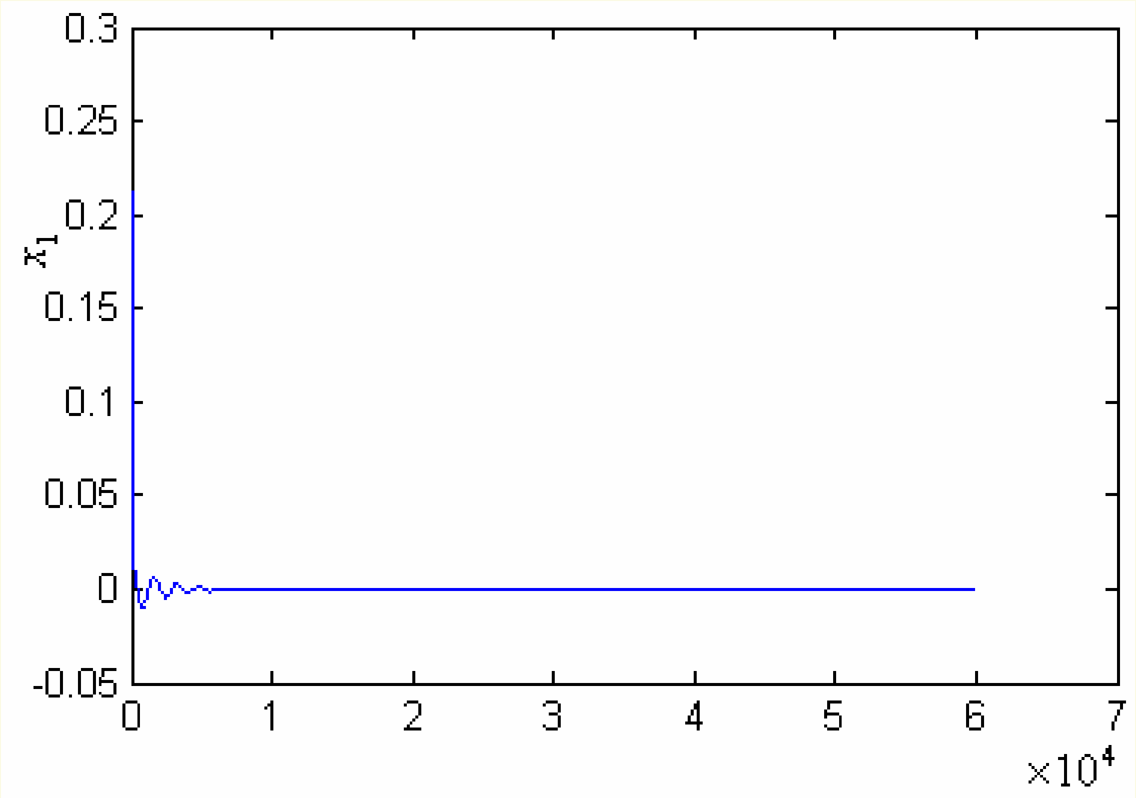
Coupling strength: $c = 22.9$

Controllers parameter:

Control gain: $k = 200$

Pinning Control of Chua-Circuit Network

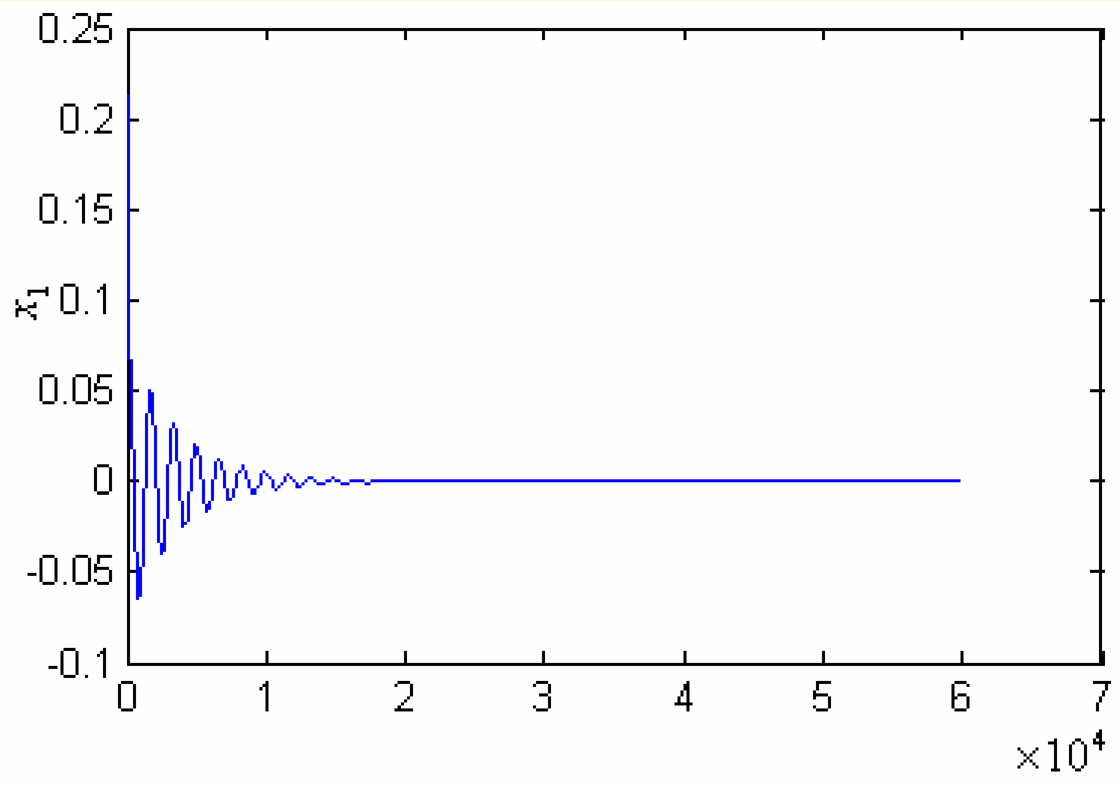
Case I: All nodes are pinned



The controlled state x1 (Time step:0.001)

Pinning Control of Chua-Circuit Network

Case II: Selectively pinned



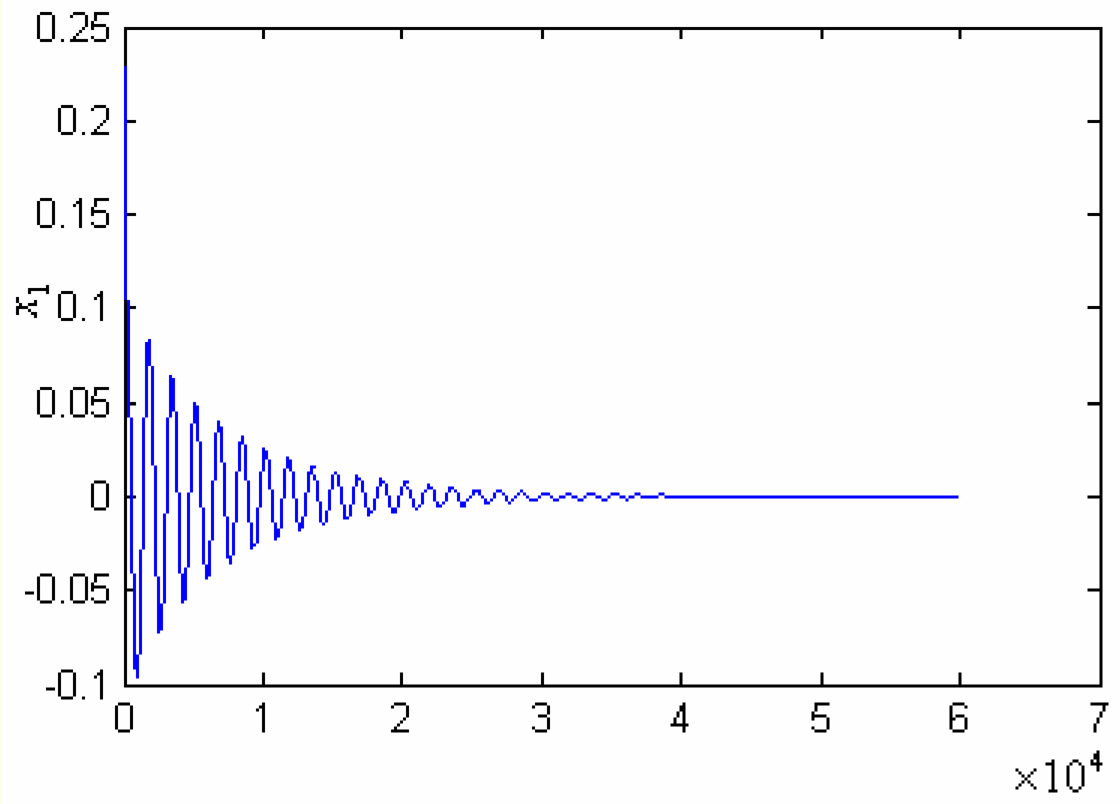
40% nodes are pinned

The controlled state x1

(Time step:0.001)

Pinning Control of Chua-Circuit Network

Case III: Randomly pinned



40% nodes are pinned

The controlled state x1

(Time step:0.001)

Recall: A Typical Scale-Free Network Model

The scale-free network model:

$$\dot{x}_i = f(x_i) + c\sum_{\substack{j=1\\ j\neq i}}^{N} a_{ij}\Gamma(x_j - x_i), i = 1,2,\cdots,N \qquad (1)$$

$$\dot{x}_i = f(x_i) + c\sum_{j=1}^{N} a_{ij}\Gamma x_j \qquad i = 1,2,\cdots,N \qquad (2)$$

Here, the coupling matrix $A = (a_{ij}) \in R^{N\times N}$ represents the coupling configuration of the entire network, which is a symmetric and irreducible matrix.

Pinning Control of Scale-Free Networks

Suppose that **nodes** $1,2,\cdots l$ are selected to be **pinned**

Then, the **controlled network** is

$$\begin{cases} \dot{x}_i = f(x_i) + c\sum_{j=1}^{N} a_{ij}\Gamma x_j + u_i,\ i = 1,2,\cdots,l \\ \dot{x}_i = f(x_i) + c\sum_{j=1}^{N} a_{ij}\Gamma x_j,\ i = l+1, l+2,\cdots,N \end{cases} \quad (3)$$

Pinning Control of Scale-Free Networks

Rewrite network (3) as

$$\dot{x}_i = f(x_i) + c\sum_{j=1}^{N} a_{ij}\Gamma x_j + b_{ii}u_i, i = 1,2,\cdots,N \qquad (4)$$

Here, diagonal element $b_{ii} = 1$, if node i is pinned; otherwise, $b_{ii} = 0$.

Apply time-delay feedback control

$$u_i = k_i\Gamma(x_i(t) - x_i(t-\tau)) \qquad (5)$$

Here, k_i is the constant control gain and τ is the constant delayed time.

Pinning Control of Scale-Free Networks

Then, network (4) becomes

$$\dot{x}_i = f(x_i) + c\sum_{j=1}^{N} a_{ij}\Gamma x_j + b_{ii}k_i\Gamma(x_i(t) - x_i(t-\tau)), i = 1,2,\cdots,N$$

Let $x_i(t) = \overline{x} + e_i(t)$, so that

$$\dot{e}_i = f(\overline{x} + e_i(t)) - f(\overline{x}) + c\sum_{j=1}^{N} a_{ij}\Gamma e_j + b_{ii}k_i\Gamma(e_i(t) - e_i(t-\tau)),$$

$$i = 1,2,\cdots,N \qquad (6)$$

Pinning Control of Scale-Free Networks

Lemma:

The synchronization error system (6) is asymptotically stable about its zero equilibrium if the following linear system is asymptotically stable about the zero equilibrium:

$$\dot{e}_i = (J(t) + b_{ii}k_i\Gamma)e_i(t) + c\sum_{j=1}^{N} a_{ij}\Gamma e_j - b_{ii}k_i\Gamma e_i(t-\tau),$$

$$i = 1, 2, \cdots, N$$

Pinning Control of Scale-Free Networks

Theorem: **The controlled network (4) will be controlled to the target asymptotically if there exist symmetrical and positive-definite matrices** $W, X, Z \in R^{n\times n}$ **such that the following LMI holds:**

$$M = \begin{bmatrix} \hat{A} & ca_{i1}\Gamma W & \cdots & ca_{iN}\Gamma W & 0 & \cdots & -b_{ii}\Gamma X & \cdots & 0 \\ ca_{i1}W\Gamma & Z & & & & & & & \\ \vdots & & \ddots & & & & & & \\ ca_{iN}W\Gamma & & & Z & & & & & \\ 0 & & & & -Z & & & & \\ \vdots & & & & & \ddots & & & \\ -b_{ii}X\Gamma & & & & & & & & \\ \vdots & & & & & & & & \\ 0 & & & & & & & & -Z \end{bmatrix} < 0$$

Here: $\hat{A} = WJ^T + JW + b_{ii}X\Gamma + b_{ii}\Gamma X$ and J involves the control gains

Pinning Control of Scale-Free Networks

Proof: Construct a Lyapunov functional as

$$V = \sum_{i=1}^{N} \left\{ e_i^{\,T}(t) P e_i(t) + \sum_{j=1}^{N} \int_{t-\tau}^{t} e_j^{\,T}(\sigma) R e_j(\sigma) d\sigma \right\}$$

Here, *P* and *Q* are symmetrical and positive-definite.

$$\dot{V}(e_1, e_2, \cdots e_N) = \sum_{i=1}^{N} \left\{ e_i^{\,T}(t) \left(\left(J^{T}(t) + b_{ii} k_{ii} \Gamma \right) P + P \left(J(t) + b_{ii} k_{ii} \Gamma \right) \right) e_i(t) \right.$$

$$+ 2c \left[\sum_{j=1}^{N} a_{ij} \Gamma e_j(t) \right]^{T} P e_i(t) - 2 b_{ii} k_i e_i^{\,T}(t-\tau) \Gamma P e_i(t)$$

$$\left. + \sum_{j=1}^{N} e_j^{\,T}(t) \mathrm{R} e_j(t) - \sum_{j=1}^{N} e_j^{\,T}(t-\tau) \mathrm{R} e_j(t-\tau) \right\}$$

Pinning Control of Scale-Free Networks

Therefore, the derivative of $V(e_1, e_2, \cdots e_N)$ is negative if

$$M = \begin{bmatrix} A & ca_{i1}P\Gamma & \cdots & ca_{iN}P\Gamma & 0 & \cdots & -b_{ii}k_i\Gamma P & \cdots & 0 \\ ca_{i1}\Gamma P & R & & & & & & & \\ \vdots & & \ddots & & & & & & \\ ca_{iN}\Gamma P & & & R & & & & & \\ 0 & & & & -R & & & & \\ \vdots & & & & & \ddots & & & \\ -b_{ii}k_i\Gamma P & & & & & & -R & & \\ \vdots & & & & & & & \ddots & \\ 0 & & & & & & & & -R \end{bmatrix} < 0 \quad (7)$$

Pinning Control of Scale-Free Networks

Multiplying $\begin{bmatrix} P^{-1} & & \\ & \ddots & \\ & & P^{-1} \end{bmatrix}$ to both sides of (7) yields

$$M = \begin{bmatrix} \tilde{A} & ca_{i1}\Gamma P^{-1} & \cdots & ca_{iN}\Gamma P^{-1} & 0 & \cdots & -b_{ii}k_i\Gamma P^{-1} & \cdots & 0 \\ ca_{i1}P^{-1}\Gamma & P^{-1}RP^{-1} & & & & & & & \\ \vdots & & \ddots & & & & & & \\ ca_{iN}P^{-1}\Gamma & & & P^{-1}RP^{-1} & & & & & \\ 0 & & & & -P^{-1}RP^{-1} & & & & \\ \vdots & & & & & \ddots & & & \\ -b_{ii}k_iP^{-1}\Gamma & & & & & & & & \\ \vdots & & & & & & & & \\ 0 & & & & & & & & -P^{-1}RP^{-1} \end{bmatrix} < 0$$

Here, $\tilde{A} = P^{-1}(J^T(t) + b_{ii}k_i\Gamma) + (J(t) + b_{ii}k_i\Gamma)P^{-1}$

Let $W = P^{-1}, X = k_iP^{-1}, Z = P^{-1}RP^{-1}$. Then, it completes the proof of the theorem.

Pinning Control of Scale-Free Networks

Example: A coupled scale-free dynamical network

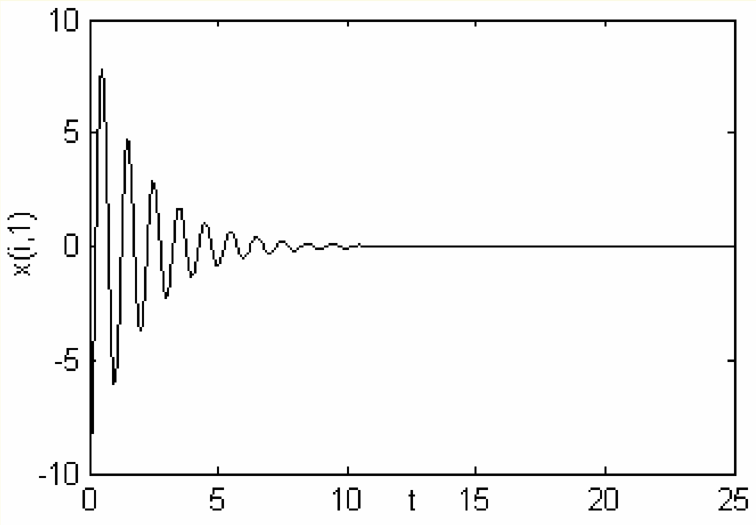
$$\dot{x}_i = \begin{pmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \\ \dot{x}_{i4} \end{pmatrix} = \begin{pmatrix} -x_{i3} - x_{i4} + c\sum_{j=1}^{N} a_{ij} x_{j1} \\ 2x_{i2} + x_{i3} + c\sum_{j=1}^{N} a_{ij} x_{j2} \\ 14x_{i1} - 14x_{i2} + c\sum_{j=1}^{N} a_{ij} x_{j3} \\ 100x_{i1} - 100x_{i4} \\ \quad + 100(\left(x_{i4}+1\right) - \left(x_{i4}-1\right)) \\ \quad + c\sum_{j=1}^{N} a_{ij} x_{j4} \end{pmatrix}.$$

$$(i = 1, 2, \cdots N)$$

Pinning Control of Scale-Free Networks

Here, network size $N = 60$, coupling strength $c = 8.246$, and number of pinning nodes $l = 15$

Selective Pinning Control: Only pin the first 15 largest-degree nodes, with control gains $k_i = 29.7603$



The controlled state x1

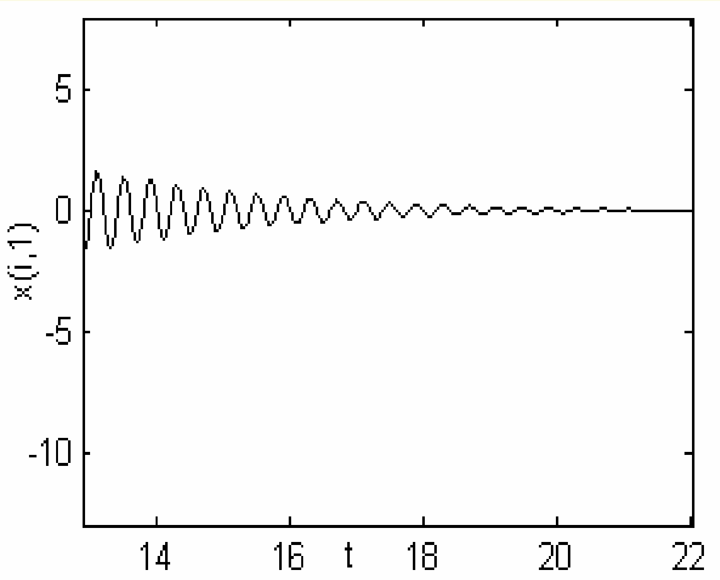
Pinning Control of Scale-Free Networks

Comparison:

Random Pinning Control: Randomly select 15 nodes.

Control gains are $k_i = 513.3709$

Much larger than the last one: $k_i = 29.7603$



And, it takes twice-long time to stabilize the network

The controlled state x1

Pinning Control of Scale-Free Networks

The selective pinning control scheme utilizes the special structures of scale-free complex networks. Therefore, it can give much better control performance than the random pinning control scheme

→ A good control strategy should utilize the structures of the complex networks

Pinning Control of Scale-Free Networks:

Possibly the Simplest Pinning Control

Apply pinning control with a constant control input:

$$u_i = -kd_i \Gamma B \tag{8}$$

Here, k is the constant control input;

$d_i = 1$ if node i is pinned; otherwise, $d_i = 0$.

Moreover, $B = [1,1,\ldots,1]^T \in R^{n\times 1}$.

Let $x_i(t) = \bar{x} + e_i(t)$, so that

$$\begin{cases} \dot{e}_i = \dfrac{\partial f(e_i)}{\partial e_i} e_i + c\sum_{j=1}^{N} a_{ij}\Gamma e_j + u_i,\ i = 1,2,\cdots,l \\ \dot{e}_i = \dfrac{\partial f(e_i)}{\partial e_i} e_i + c\sum_{j=1}^{N} a_{ij}\Gamma e_j,\ i = l+1, l+2,\cdots,N \end{cases} \tag{9}$$

Pinning Control of Scale-Free Networks

Example: Consider a coupled scale-free dynamical network consisting of Lorenz systems:

$$\dot{x}_i = \begin{pmatrix} a(x_{i2} - x_{i1}) + c\sum_{j=1}^{N} a_{ij} x_{j1} \\ cx_{i1} - x_{i1}x_{i3} - x_{i2} + c\sum_{j=1}^{N} a_{ij} x_{j2} \\ x_{i1}x_{i2} - bx_{i3} + c\sum_{j=1}^{N} a_{ij} x_{j3} \end{pmatrix}. \qquad (11)$$

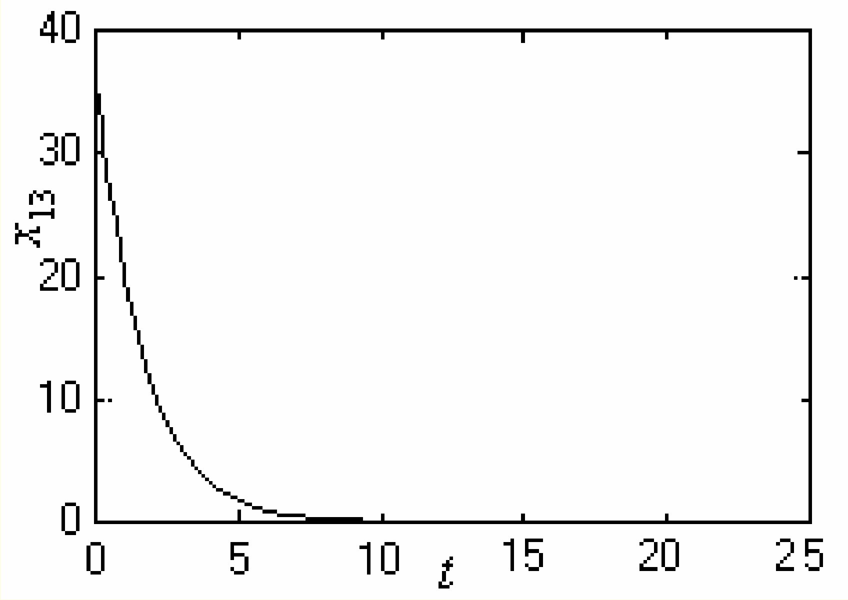
$$(i = 1, 2, \cdots N)$$

Pinning Control of Scale-Free Networks

Parameters:

$$a = 10, b = 8/3, c = 28, \ \gamma = 45, \beta = 30$$

The network size is 50. With 24 nodes being controlled, the network is well stabilized:

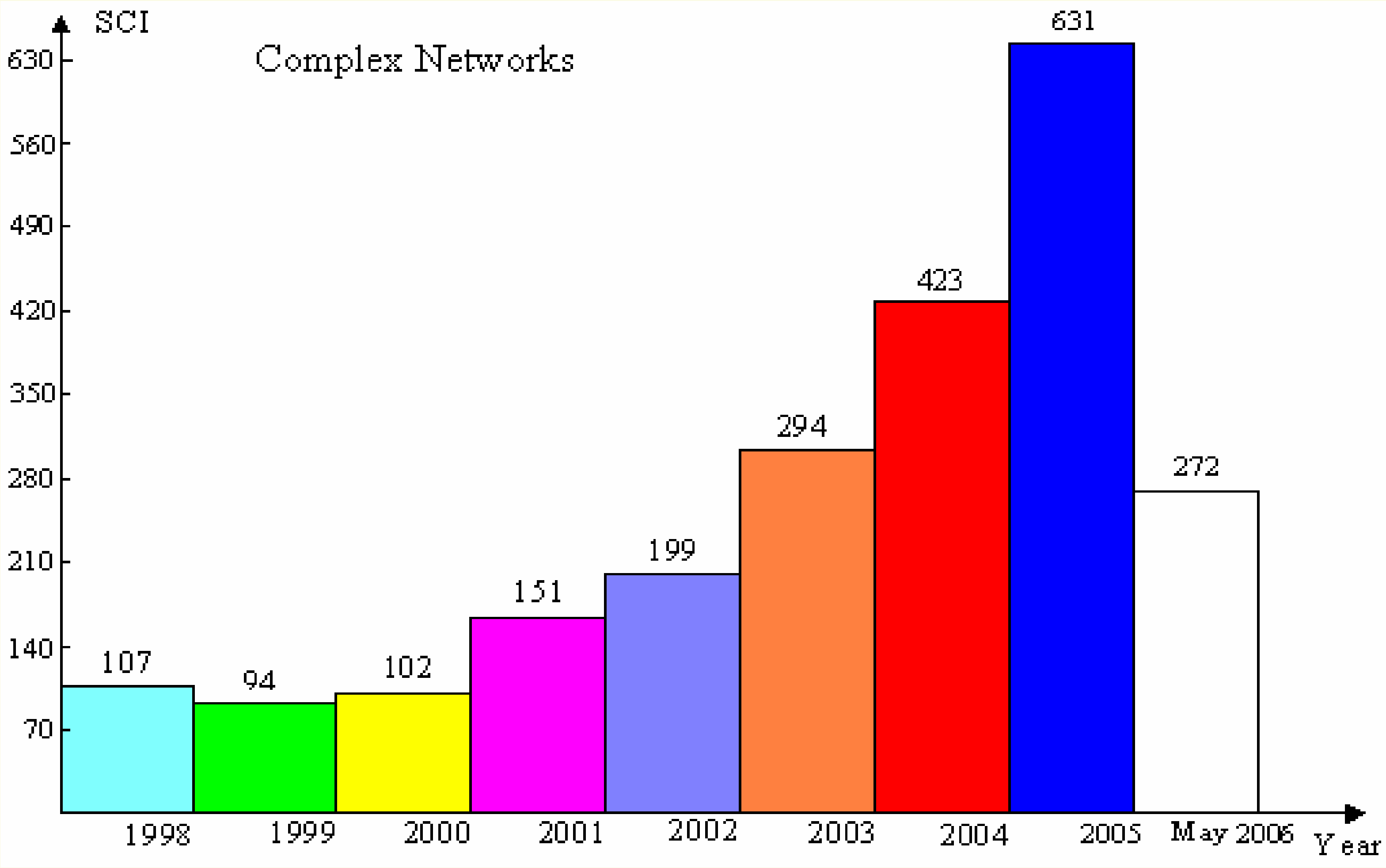


The controlled state x_3 in the largest-degree node

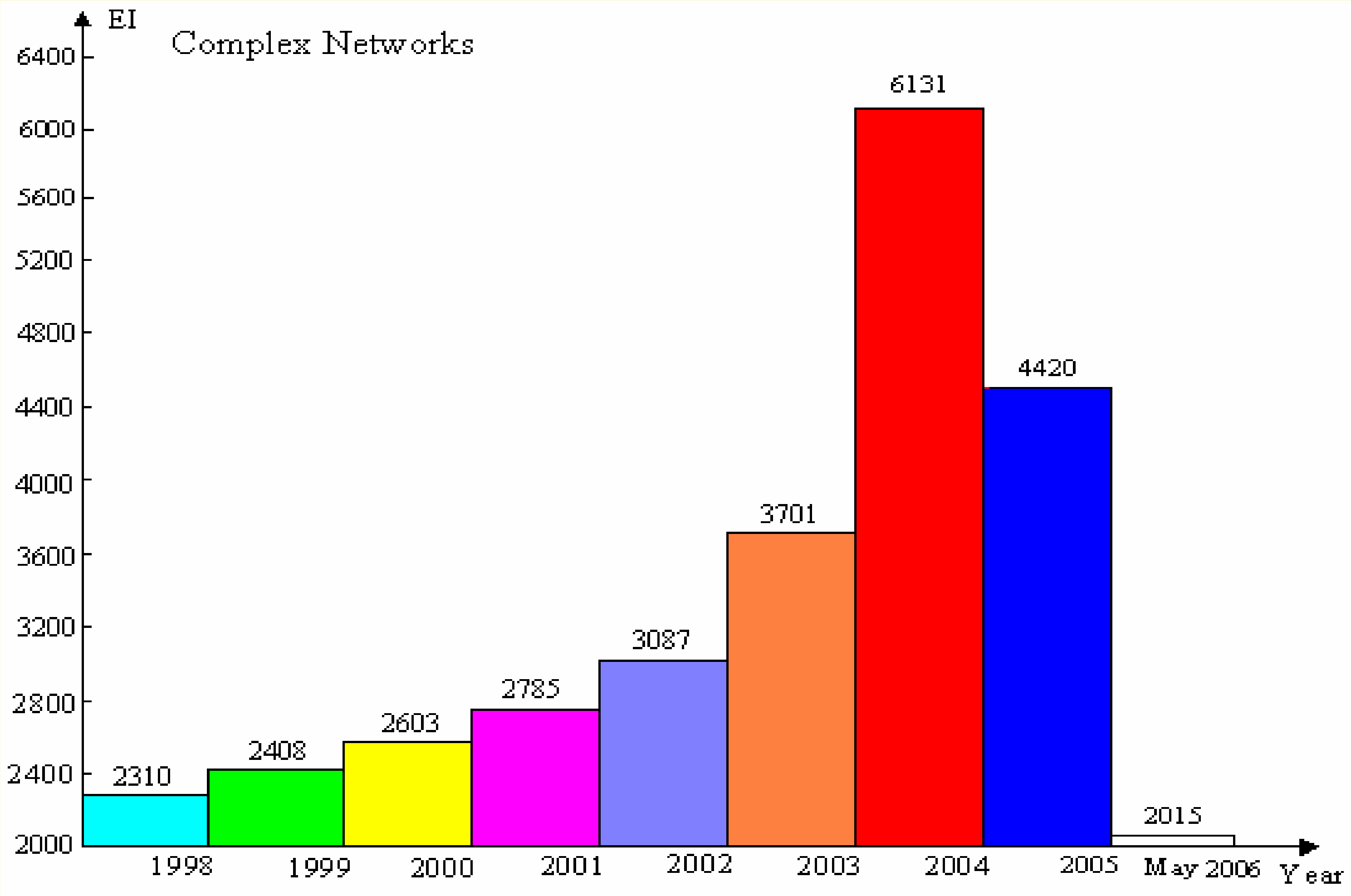
Conclusions

- Pinning is a good control strategy for scale-free dynamical networks
- Selective pinning control scheme is much more efficient than the random pinning control scheme
- A sufficient condition can be given to selective pinning of scale-free networks in terms of LMI
- Example shows that even constant pinning control input works well for some scale-free networks
- More efficient, and yet simple and cost-effective, control approaches are to be further developed

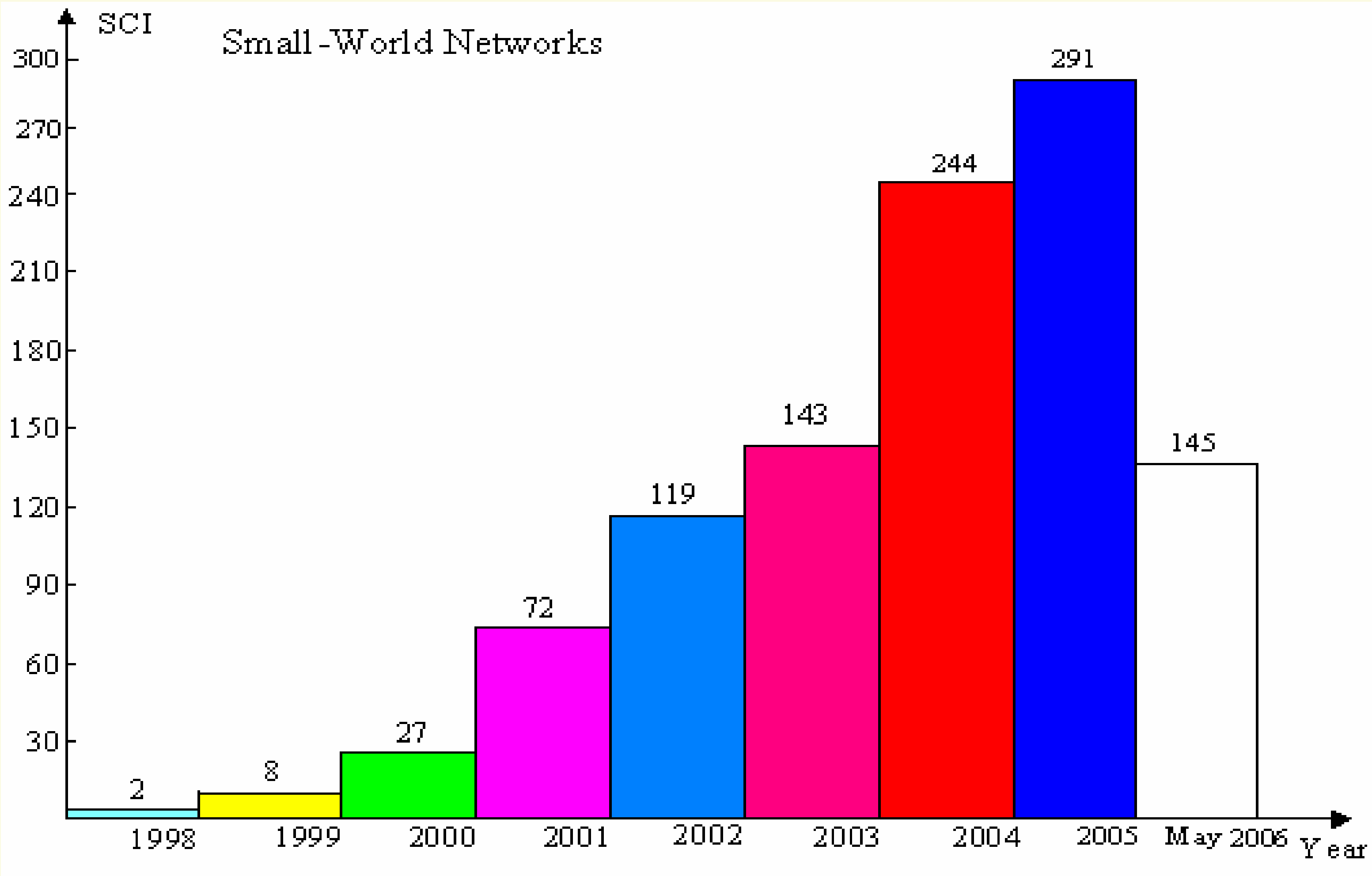
SCI papers: Complex Networks



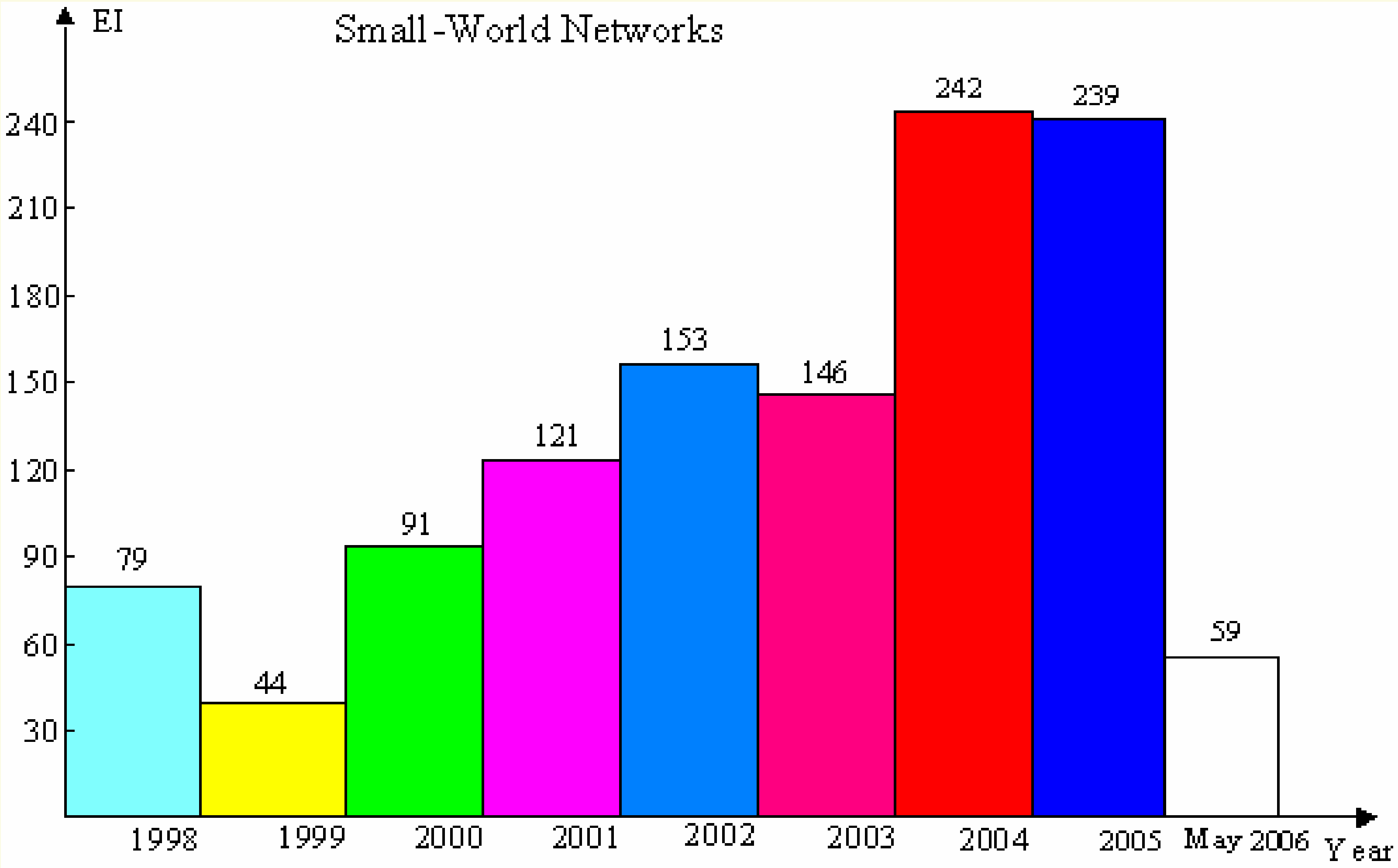
EI papers: Complex Networks



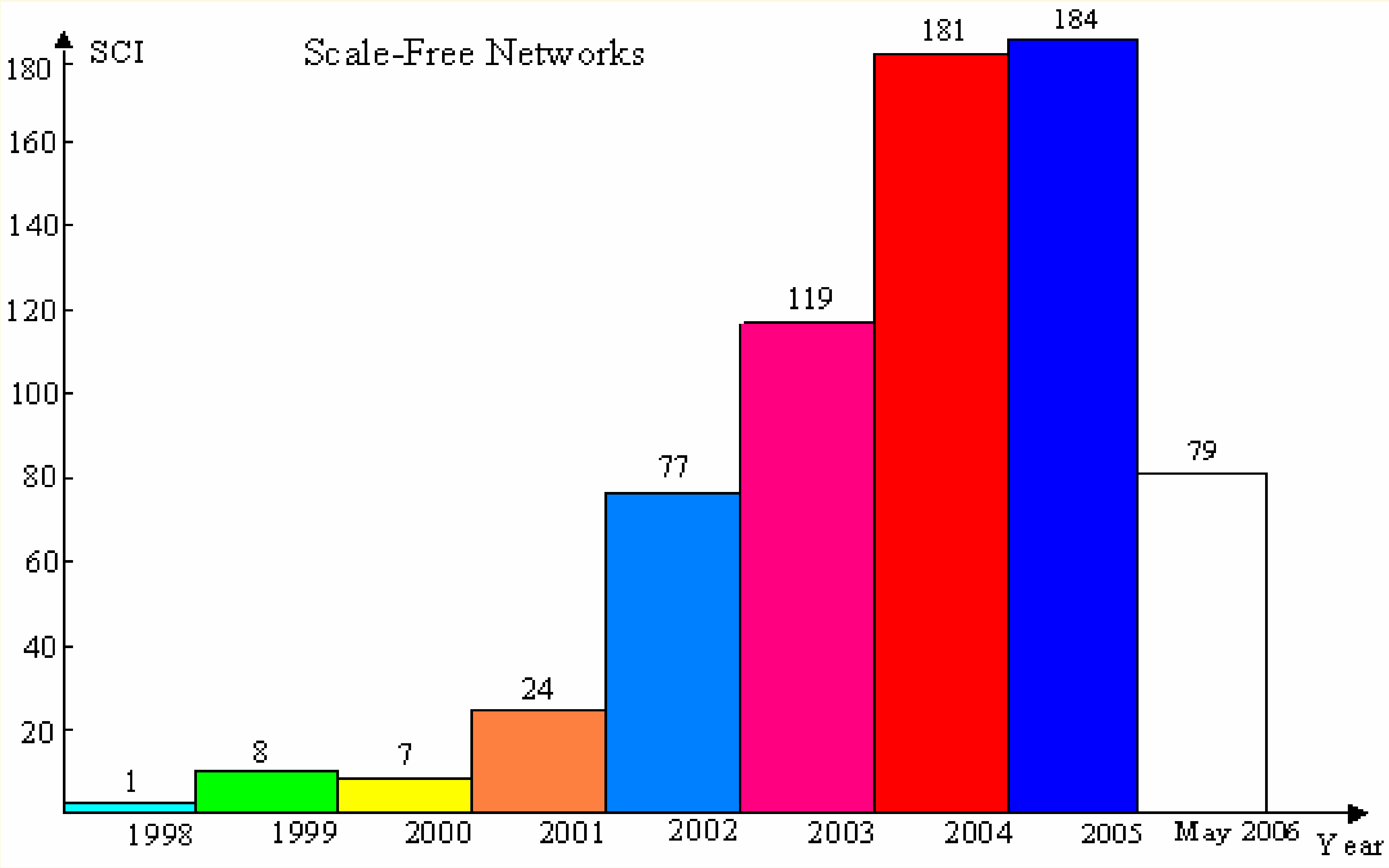
SCI papers: Small-World Networks



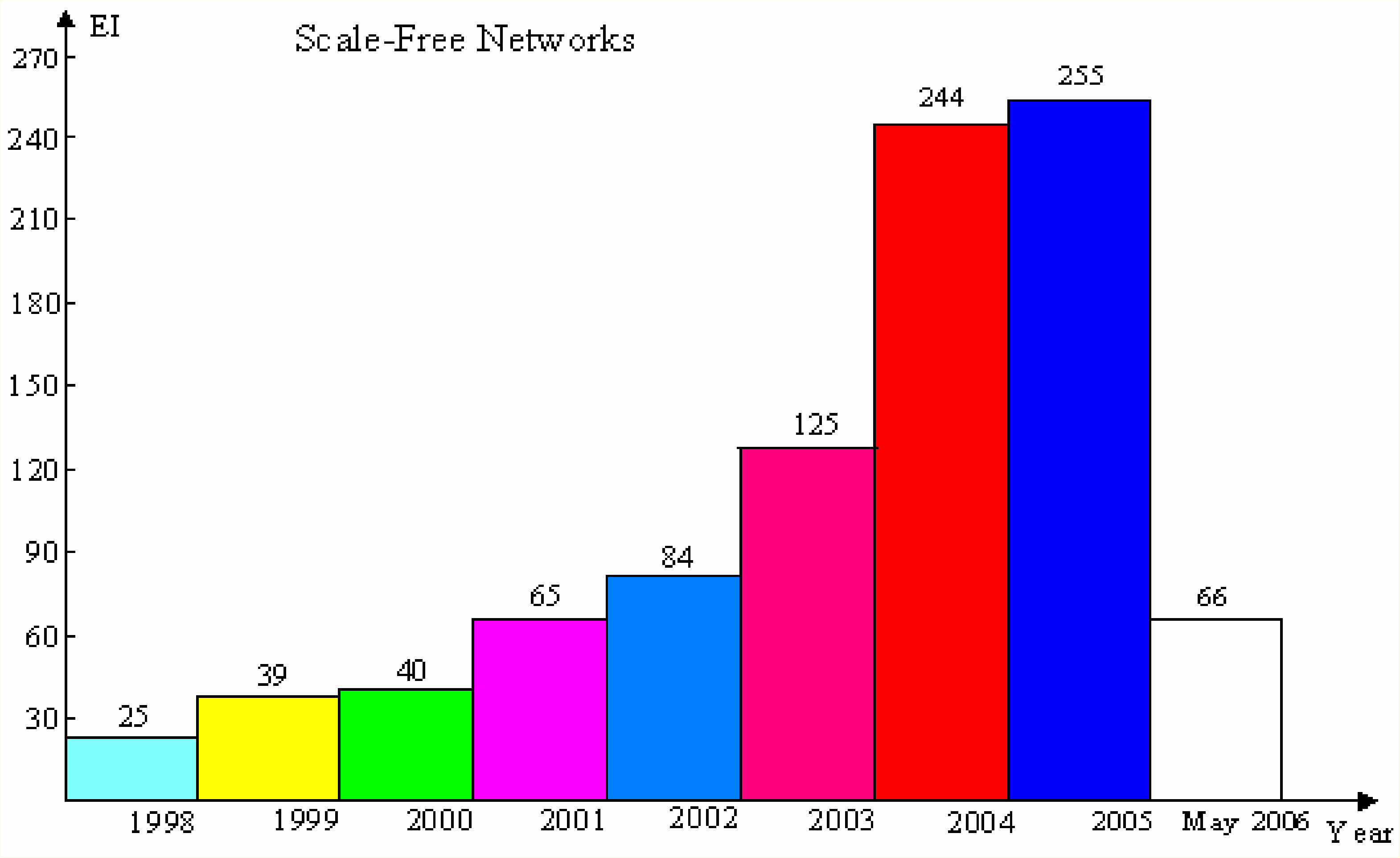
EI papers: Small-World Networks



SCI papers: Scale-Free Networks



EI papers: Scale-Free Networks



Thank You !